ON COMPACTNESS IN FUNCTIONAL ANALYSIS(1)

BY

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TO THE MEMORY OF MY TEACHER AND FRIEND

ARNOLD DRESDEN

(1882–1954)

The theorem of Arzelà and Ascoli, characterizing conditionally compact subsets of the Banach space $C(X)$ of continuous functions defined on a compact topological space X, is fundamental for much of functional analysis. Of less importance but still of interest is the question of characterizing subsets of $C(X)$ which are conditionally compact in other naturally chosen topologies, such as the weak topology of $C(X)$ as a Banach space, or the topology of pointwise convergence. This problem was considered in the case that $X=[0, 1]$ by G. Sirvint [19; 20](2). It is the purpose of this paper to treat the general case; in doing so we adopt quite a different point of view.

We shall find it convenient to make use of the notions of universal nets (here called U-nets) introduced by J. L. Kelley [13] and of quasi-uniform convergence due to C. Arzelà [1]. Since previous familiarity with these concepts is not assumed, in §§1 and 2 we state the properties that will be needed.

In order to have a wide range of applicability, we have chosen to present §3 in an abstract formulation which is specialized in later parts. It is felt that this treatment brings out clearly the rôle of quasi-uniform convergence; further, it emphasizes the duality inherent in these compactness theorems, but which is not usually made explicit. This duality was suggested by R. S. Phillips [17] and a form of it was employed systematically by V. Šmulian [21] for bounded subsets of a separable Banach space. The general formulation was made by S. Kakutani [12] for the case of uniform convergence, in much the same vein as here. The results of §3 are applied, in §4, to certain subsets of a Banach space in order to fix these ideas and to indicate how readily they permit symmetric proofs of the classical theorems of [Gantmacher](2) Schauder concerning [weakly] compact operators and their adjoints.

In §5, we prove that a necessary and sufficient condition that a collection

Presented to the Society, September 3, 1954; received by the editors January 15, 1954.

(1) This paper was written while the author was under contract with the Office of Naval Research.

(2) Numbers in brackets ordinarily refer to the citations at the end of the paper. Names, words, or phrases in brackets indicate a double reading is intended. Names, words, or phrases in parentheses always indicate a parenthetical remark.

of continuous functions on a compact space X be conditionally compact in the topology of pointwise convergence is that their values at each point be bounded and that they satisfy a certain "quasi-equicontinuity" property. If this topology is replaced by the weak topology of $C(X)$, then it is seen in §6 that the set of functions must be uniformly bounded and quasi-equicontinuous. Other results closely related to theorems of Grothendieck [11] and Eberlein [7] are given. In §7, we drop the requirement of the compactness of X and find similar conditions; in §8 analytic functions are discussed briefly. We state some extensions of results due to Sirvint and Gelfand in §§9 and 10. Although most of the results in these two sections are known, at least in the separable case, we have chosen to include them since they are ready consequences of our previous discussion and since there seems to be some advantage in collecting the weak and strong results in one place.

Whenever possible, we have tried to arrange the material so that weak (or pointwise) results and strong results may be contrasted and compared. This has been done at the cost of restating known theorems (e.g., the Arzelà-Ascoli theorem), usually without detailed proof, and often in the form of optional readings in a given statement. We have done this in the hope of emphasizing the parallelism of the two cases, and of enhancing the reference value of this work for the reader.

1. **Universal nets.** Since we will make free use of the concept of a net and particularly that of a U-net or universal net, we recall here the definitions and fundamental properties. The reader is referred to the work of Kelley [13] for proofs and additional results.

A *directed set* $A=\{\alpha\}$ is a set which is partially ordered by some ordering, written "$\leqq$," with the additional property that, given α_1, $\alpha_2 \in A$ there exists an $\alpha \in A$ with $\alpha_i \leqq \alpha$. A *net* (or directed system) in a set X is a function from a directed set to X; we shall always write the argument as a subscript, thus (x_α) is a net in X. We say that a net (x_α) in X is *ultimately in* $S \subseteq X$ if there exists an index α_S such that if $\alpha \geqq \alpha_S$ then $x_\alpha \in S$. A *net* (x_α) *converges to* x_0, in symbols $x_\alpha \rightarrow x_0$, if it is ultimately in every subset S which contains x_0. If X is a topological space it is customary to restrict the sets S to neighborhoods of x_0.

1.1. Definition. A net (x_α) in a set X is said to be *universal*, or to be a U-*net*, if for every $S \subseteq X$, (x_α) is ultimately in either S or its complement S'.

1.2. Definition. If $(x_\alpha)_{\alpha \in A}$ is a net in X, then a net $(x_\beta)_{\beta \in B}$ in X is said to be a *subnet* of (x_α) provided that there is a function $\pi: B \rightarrow A$ with the properties:

(1) $x_\beta = x_{\pi(\beta)}$, $\beta \in B$;

(2) Given $\alpha_0 \in A$, there exists a $\beta_0 = \beta_0(\alpha_0)$ such that if $\beta \geqq \beta_0$ then $\pi(\beta) \geqq \alpha_0$.

It is readily seen that if X and Y are topological spaces and $f: X \rightarrow Y$, then f is continuous if and only if it maps convergent nets in X into convergent nets in Y. Further, if a net converges to a point, every subnet converges to the same limit. We also require:

1.3. LEMMA. *If X and Y are arbitrary sets and $f:X\to Y$ then f maps U-nets in X into U-nets in Y.*

1.4. LEMMA. *Let X and Y be topological spaces and $f:X\to Y$. If $x_\alpha\to x_0$, but the net $(f(x_\alpha))$ does not converge to $f(x_0)$, then there exists a neighborhood V of $f(x_0)$ and a subnet (x_β) of (x_α) such that $f(x_\beta)\notin V$, for all β.*

The following theorem is considerably deeper; it will be important for our applications:

1.5. THEOREM. *If X is an arbitrary set, then every net in X has a subnet which is a U-net.*

2. **Quasi-uniform convergence.** The notion of uniform convergence is deservedly well known; that of quasi-uniform convergence, although introduced for sequences of functions by Arzelà [1] in 1883, is much less common. Nevertheless, we shall see in later sections that quasi-uniform convergence is remarkably well suited for questions of pointwise convergence and weak topologies. The present section consists of definitions and immediate applications of uniform and quasi-uniform convergence that will be needed.

2.1. DEFINITION. A net (f_α) of (scalar-valued) functions on an arbitrary set X is said to *converge to f_0 uniformly on X*, if for every $\epsilon>0$ there exists an $\alpha_0=\alpha_0(\epsilon)$ such that if $\alpha\geqq\alpha_0$ then $|f_\alpha(x)-f_0(x)|<\epsilon$, $x\in X$.

2.2. DEFINITION. A net (f_α) of (scalar-valued) functions on an arbitrary set X is said to *converge to f_0 quasi-uniformly on X*, if $f_\alpha(x)\to f_0(x)$ for all $x\in X$ and if, for every $\epsilon>0$ and α_0, there exists a finite number of indices $\alpha_1, \cdots, \alpha_n\geqq\alpha_0$ such that for each $x\in X$ at least one of the following inequalities holds:

$$|f_{\alpha_i}(x) - f_0(x)| < \epsilon, \qquad i = 1, \cdots, n.$$

We shall say that a net of functions converges [quasi-] uniformly on X if there is a function to which the net converges [quasi-] uniformly on X. It will be obvious to the reader what is meant by [quasi-] uniform convergence on a subset of X. It is immediate that:

2.3. LEMMA. *If a net of functions converges uniformly on X, then every subnet also converges uniformly on X.*

2.4. LEMMA. *If a net of functions converges on X, and if some subnet converges quasi-uniformly on X, then the net converges quasi-uniformly on X.*

It can be seen that the uniform convergence of a subnet does not imply the uniform convergence of the original net, nor does the quasi-uniform convergence of a net of functions imply the quasi-uniform convergence of a proper subnet.

The importance of quasi-uniform convergence stems from the fact that it is necessary and sufficient for the limit of a net of continuous functions to be continuous.

2.5. THEOREM (ARZELÀ). *If a net of continuous functions on a topological space X converges to a continuous limit, then the convergence is quasi-uniform on every compact subset of X. Conversely, if the net converges quasi-uniformly on a subset of X, the limit is continuous on this subset.*

Proof. Let f_0 be the limit of a net $(f_\alpha)_{\alpha \in A}$ of continuous functions on X. If f_0 is continuous on X, then given $\epsilon > 0$, $\alpha_0 \in A$, and $y \in X$, there is an $\alpha(y) \geqq \alpha_0$ such that $|f_{\alpha(y)}(y) - f_0(y)| < \epsilon$. Let $N(y) = \{x;\ |f_{\alpha(y)}(x) - f_0(x)| < \epsilon\}$; since f_0 and the f_α are continuous, $N(y)$ is an open set containing y. If C is a compact subset of X, only a finite number of sets $N(y_i)$, $y_i \in C$, $i = 1, \cdots, n$, are required to cover C. The indices $\alpha(y_i)$, $i = 1, \cdots, n$, then satisfy the definition of quasi-uniform convergence.

Conversely, suppose that (f_α) converges to f_0 quasi-uniformly on a subset $B \subseteq X$. Given $\epsilon > 0$, $x_0 \in B$, there is an α_0 such that if $\alpha \geqq \alpha_0$ then $|f_\alpha(x_0) - f_0(x_0)| < \epsilon$. Select $\alpha_1, \cdots, \alpha_n \geqq \alpha_0$ as described in the definition of quasi-uniform convergence and let $N_i = \{x \in B;\ |f_{\alpha_i}(x) - f_{\alpha_i}(x_0)| < \epsilon\}$. The sets N_i are open in B and contain x_0; hence $N = \bigcap_{i=1}^{n} N_i$ is open in B and contains x_0. Now for any $x \in N \cap B$ and proper choice of i, we have

$$|f_0(x) - f_0(x_0)| \leqq |f_0(x) - f_{\alpha_i}(x)| + |f_{\alpha_i}(x) - f_{\alpha_i}(x_0)| + |f_{\alpha_i}(x_0) - f_0(x_0)| < 3\epsilon.$$

Thus f_0 is continuous at the arbitrary point $x_0 \in B$.

2.6. COROLLARY. *On a compact topological space, the limit of a net of continuous functions is continuous if and only if the convergence is quasi-uniform.*

2.7. COROLLARY. *Let X be a compact topological space, and suppose that a net (f_α) of continuous functions on X converges on X to a continuous limit f_0. Then f_0 is continuous in any topology on X in which all the f_α are continuous.*

Proof. By Corollary 2.6, the net (f_α) converges quasi-uniformly on X. Thus f_0 is continuous with the weakest topology on X which makes all the f_α continuous, and *a fortiori* with any stronger topology on X.

A net (f_α) of real-valued functions on a set X is said to be monotone increasing if $\alpha \leqq \beta$ implies $f_\alpha(x) \leqq f_\beta(x)$, $x \in X$. The definition of monotone decreasing is similar.

2.8. THEOREM (DINI). *If a monotone net of continuous functions on a topological space X converges to a continuous limit, then the convergence is uniform on every compact subset of X. Thus a monotone net of continuous functions on a* *compact set converges to a continuous limit if and only if the convergence is uniform.*

Proof. Suppose that (f_α) is monotone increasing; since the limit function f_0 is continuous, the convergence is quasi-uniform on every compact subset $C \subseteq X$. Thus given $\epsilon > 0$ and α_0 there exist indices $\alpha_1, \cdots, \alpha_n \geqq \alpha_0$ such that for each $x \in C$, one of the following inequalities holds:

$$[*] \qquad f_0(x) < f_{\alpha_i}(x) + \epsilon, \qquad i = 1, \cdots, n.$$

Let α' be any index with $\alpha' \geqq \alpha_1, \cdots, \alpha_n$. Hence if $\alpha \geqq \alpha'$ it follows from [*] and the monotoneity of (f_α) that

$$f_0(x) < f_\alpha(x) + \epsilon, \qquad x \in C,$$

from which the uniform convergence on C is immediate.

The following theorem will be needed and can be proved by standard arguments. It is interesting to note that the corresponding statement for a quasi-uniformly net of functions is not valid, a fact that will cause some inconvenience.

2.9. THEOREM. *Let (f_α) be a net of continuous functions on a topological space X which converges uniformly on a set Z dense in X. Then (f_α) converges uniformly on all of X.*

2.10. REMARK. The preceding definitions and results in 2.1–2.7 remain valid for functions having their values in an arbitrary uniform space, with only obvious notational modifications. In Theorem 2.9, completeness is required in the range space.

3. **Weak and strong pairing.** In this section we derive some theorems exhibiting a duality between the convergence of points in a set and convergence of functions on the set—these results will be applied to more concrete cases in later sections. There are essentially two types of results: one relating to quasi-uniform convergence, the other to uniform convergence. Similar discussions can be found in Kakutani [12], Phillips [17], and Šmulian [21].

3.1. DEFINITION. Two abstract sets X and Y are said to be *paired* if there is a scalar-valued(3) function h defined on $X \times Y$ such that:

(1) If $x \in X$, there is an $M_x > 0$ such that $|h(x, y)| < M_x$ for all $y \in Y$.

(2) If $y \in Y$, there is an $M_y > 0$ such that $|h(x, y)| < M_y$ for all $x \in X$.

In such a case we shall say that the sets X and Y are *paired by the function h.*

If X and Y are paired by h, then it is clear that each $x \in X$ can be regarded in a natural way as a bounded function ξ_x defined on Y by $\xi_x(y)$

(3) It will be seen that h may be permitted to take its values in any Hausdorff uniform space if (1) and (2) are replaced by the assumption that $h(x, Y)$ and $h(X, y)$ are conditionally compact for each $x \in X$, $y \in Y$.

$=h(x, y)$, $y\in Y$. Similarly each $y\in Y$ gives rise to a bounded scalar-valued function η_y on X. Our first task is to extend these functions to be defined on larger sets.

Let $I_y=h(X, y)=\{h(x, y); x\in X\}$, so from (2), $\bar{I}_y$ is a compact set of scalars. We embed X in the product space $P_{y\in Y}\bar{I}_y$ by the mapping $\lambda: x\to[h(x, y)]_{y\in Y}=[\xi_x(y)]_{y\in Y}$. From the theorem of Tychonoff, the space $P_{y\in Y}\bar{I}_y$ is a compact Hausdorff space in the usual product topology.

3.2. DEFINITION. By $\mathfrak{X}$ we denote the closure of $\lambda(X)$ in the space $P_{y\in Y}\bar{I}_y$ equipped with the product topology. We also refer to this topology as the *Y-topology* of $\mathfrak{X}$, thus emphasizing that it is generated by elements of Y.

Since each element $\mathfrak{x}\in\mathfrak{X}$ is a collection $[\sigma_y]_{y\in Y}$ of scalars it is natural to extend h to a function h_1 on $\mathfrak{X}\times Y$ by setting

$$h_1(\mathfrak{x}, y) = \text{proj}_y(\mathfrak{x}), \qquad \mathfrak{x}\in\mathfrak{X}, \ y\in Y.$$

Since there is no ambiguity, we shall permit ourselves to write $h_1(x, y)$ instead of $h_1(\lambda(x), y)$. Further, by definition of $\mathfrak{X}$, each $\mathfrak{x}\in\mathfrak{X}$ is the limit of a net $(\lambda(x_\alpha))$ of elements in $\lambda(X)$ and for each $y\in Y$, the nets $(h(x_\alpha, y))$ of scalars converge to $h_1(\mathfrak{x}, y)$. For each fixed $y\in Y$, there is a continuous function η_y defined on $\mathfrak{X}$ by $\eta_y(\mathfrak{x})=h_1(\mathfrak{x}, y)$ and which coincides on X with the function previously denoted by this symbol.

Although $\lambda(X)\subseteq\mathfrak{X}$, it is not strictly true that X can be considered to be a subset of $\mathfrak{X}$, since it is quite likely that λ is not one-to-one. Nevertheless, if we equip X with the topology generated by the subbasic neighborhoods $N(x_0; y, \epsilon)=\{x; |h(x, y)-h(x_0, y)|<\epsilon\}$, then λ is a continuous and open mapping of X onto $\lambda(X)\subseteq\mathfrak{X}$. This topology in X, which we shall call the *Y-topology* of X, has the disadvantage that it is not Hausdorff unless λ is one-to-one, but it is the most natural one for the study of the functions generated by Y since it is the weakest topology on X which renders continuous all the functions generated by Y. Further, any stronger topology on X will destroy the openness of the map λ onto its range.

We summarize these comments:

3.3. LEMMA. *The set $\mathfrak{X}$ is a compact Hausdorff space which contains the image $\lambda(X)$ of X as a dense subset. With the Y-topology on X, λ is a continuous and open map onto $\lambda(X)$. The set $\Upsilon=\{\eta_y; y\in Y\}$, where $\eta_y(\mathfrak{x})=h_1(\mathfrak{x}, y)$, $\mathfrak{x}\in\mathfrak{X}$, is a collection of continuous functions on $\mathfrak{X}$, the Y-topology on $\mathfrak{X}$ being the weakest in which all the functions in Υ are continuous. If $x\in X$, then $\eta_y(x)=h_1(x, y)=h(x, y)$.*

In exactly the same manner, we let $I_x=h(x, Y)$ and define the map $\mu: y\to[h(x, y)]_{x\in X}$ sending Y into $P_{x\in X}\bar{I}_x$. By $\mathfrak{Y}$ we denote the closure of $\mu(Y)$ in the space $P\bar{I}_x$ and by h_2 the extension of h to the set $X\times\mathfrak{Y}$. The collection of all functions $\xi_x(\mathfrak{y})=h_2(x, \mathfrak{y})$ is represented by Ξ. Thus each $x\in X$ gives rise to a continuous function ξ_x on the compact Hausdorff space $\mathfrak{Y}$.

We shall use the same (or corresponding) terminology and conventions for $\mathfrak{Y}$ and Ξ that we have introduced for $\mathfrak{X}$ and Υ.

Since the set X gives rise to a collection Ξ of continuous functions on the compact space $\mathfrak{Y}$, and also via the embedding $\lambda(X)$, to a dense subset of the compact space $\mathfrak{X}$, it is reasonable to ask whether $\mathfrak{X}$ can be represented as a family of continuous functions on $\mathfrak{Y}$. Since the X-topology on $\mathfrak{Y}$ is the weakest in which all the functions of Ξ are continuous, one would expect that, in general, a strengthening of the X-topology of $\mathfrak{Y}$ would be required if new functions generated by points in $\mathfrak{X}$ are to be continuous. Such strengthening might destroy the compactness of $\mathfrak{Y}$. One is led, therefore, to restrict the mode of convergence of a net $(\lambda(x_\alpha))\rightarrow\mathfrak{x}$ in $\mathfrak{X}$, rather than to enrich the topology of $\mathfrak{Y}$. At the same time, the symmetry makes it more or less evident that $\mathfrak{Y}$ gives rise to a family of continuous functions on $\mathfrak{X}$ if and only if $\mathfrak{X}$ has a similar realization on $\mathfrak{Y}$.

Let (x_α) be a net in X, then $(\lambda(x_\alpha))$ converges to $\mathfrak{x}$ in the Y-topology of $\mathfrak{X}$ if and only if $h(x_\alpha, y)\rightarrow h_1(\mathfrak{x}, y)$ for each $y\in Y$, or if and only if $\xi_{x_\alpha}(y)\rightarrow h_1(\mathfrak{x}, y)$ for each $y\in Y$. If (x_α) is a U-net in X, then for each $\mathfrak{y}\in\mathfrak{Y}$, the net $(h_2(x_\alpha, \mathfrak{y}))$ is a U-net of bounded scalars and thus converges to a unique limit point. Hence if (x_α) is a U-net with $\lambda(x_\alpha)\rightarrow\mathfrak{x}$, then the net (ξ_{x_α}) of continuous functions on $\mathfrak{Y}$ converges at every point of $\mathfrak{Y}$; we denote this limit function by $\xi_\mathfrak{x}$ and observe that

$$\xi_\mathfrak{x}(y) = h_1(\mathfrak{x}, y) = \eta_y(\mathfrak{x}), \qquad y \in Y.$$

It is a consequence of Corollary 2.6 that $\xi_\mathfrak{x}$ is a continuous function on $\mathfrak{Y}$ if and only if the convergence of (ξ_{x_α}) to $\xi_\mathfrak{x}$ is quasi-uniform on $\mathfrak{Y}$, or equivalently, if the convergence of $(h_2(x_\alpha, \mathfrak{y}))$ to its limit is quasi-uniform on $\mathfrak{Y}$. Thus we have obtained the equivalence of statements (1) and (3) of the next theorem. The equivalence of (2) and (4) is similar.

It should be observed that for $\mathfrak{x}\in\mathfrak{X}$ we have defined the function $\xi_\mathfrak{x}$ on $\mathfrak{Y}$ by means of a particular net (x_α) in X. It is by no means evident that this definition is always independent of the net (x_α). However, in the case we are interested in, this independence does hold as we shall see below.

If (x_α) is a U-net in X, then $\lambda(x_\alpha)\rightarrow\mathfrak{x}$ for some $\mathfrak{x}\in\mathfrak{X}$. We agree that the statement that (x_α) converges quasi-uniformly on $\mathfrak{Y}$ means that the net (ξ_{x_α}) of functions converges quasi-uniformly to $\xi_\mathfrak{x}$ on $\mathfrak{Y}$. With this convention we state:

3.4. Theorem. *The following statements are equivalent:*
(1) *Every U-net in X converges quasi-uniformly on $\mathfrak{Y}$;*
(2) *Every U-net in Y converges quasi-uniformly on $\mathfrak{X}$;*
(3) *If $\mathfrak{x}\in\mathfrak{X}$, then $\xi_\mathfrak{x}$ is continuous with the X-topology on $\mathfrak{Y}$;*
(4) *If $\mathfrak{y}\in\mathfrak{Y}$, then $\eta_\mathfrak{y}$ is continuous with the Y-topology on $\mathfrak{X}$.*

Proof. First observe that if $x\in X$, $y\in Y$, then

[*] $$\xi_x(y) = h(x, y) = \eta_y(x).$$

Now suppose that (4) holds and that (x_α) is a U-net with $\lambda(x_\alpha)\to\mathfrak{x}$. We shall show that $\xi_{\mathfrak{x}}$ is continuous at an arbitrary point $\mathfrak{y}\in\mathfrak{Y}$. For, let (y_β) be a net such that $\mu(y_\beta)\to\mathfrak{y}$ in the X-topology of $\mathfrak{Y}$. By Theorem 1.5, it is no loss of generality to suppose that (y_β) is a U-net. It follows that (η_{y_β}) converges on $\mathfrak{X}$ to a function $\eta_{\mathfrak{y}}$ which is continuous by (4). Hence $\eta_{\mathfrak{y}}(x_\alpha)\to\eta_{\mathfrak{y}}(\mathfrak{x})$. Since $\xi_{\mathfrak{x}}$ is the pointwise limit of (ξ_{x_α}) on $\mathfrak{Y}$ and $\mathfrak{y}=\lim_\beta \mu(y_\beta)$ it follows from [*] and the continuity of $\eta_{\mathfrak{y}}$ and ξ_{x_α} that

$$\begin{aligned}\xi_{\mathfrak{x}}(\mathfrak{y}) &= \lim_\alpha \xi_{x_\alpha}(\mathfrak{y}) = \lim_\alpha \lim_\beta \xi_{x_\alpha}(y_\beta)\\ &= \lim_\alpha \lim_\beta \eta_{y_\beta}(x_\alpha) = \lim_\alpha \eta_{\mathfrak{y}}(x_\alpha) = \eta_{\mathfrak{y}}(\mathfrak{x}).\end{aligned}$$

Now for each β,

$$\xi_{\mathfrak{x}}(y_\beta) = \lim_\alpha \xi_{x_\alpha}(y_\beta) = \lim_\alpha \eta_{y_\beta}(x_\alpha) = \eta_{y_\beta}(\mathfrak{x}).$$

But since (η_{y_β}) converges to $\eta_{\mathfrak{y}}$ on $\mathfrak{X}$, we have

$$\xi_{\mathfrak{x}}(\mathfrak{y}) = \eta_{\mathfrak{y}}(\mathfrak{x}) = \lim_\beta \eta_{y_\beta}(\mathfrak{x}) = \lim_\beta \xi_{\mathfrak{x}}(y_\beta),$$

so that $\xi_{\mathfrak{x}}$ is continuous at $\mathfrak{y}$.

The fact that, under these conditions, the function $\xi_{\mathfrak{x}}$ is independent of the choice of the net used to define it follows from the observation that $\xi_{\mathfrak{x}}(y)=\eta_y(\mathfrak{x})$ for all $y\in Y$ and the density of $\mu(Y)$ in $\mathfrak{Y}$.

3.5. Definition. If X and Y are two sets which are paired by the function h and satisfy the conditions of Theorem 3.4, we say that X and Y are *weakly paired by* h.

The interest in weakly paired sets is that in this case the set Ξ of continuous functions on $\mathfrak{Y}$ is conditionally compact in the Y-topology, and similarly the set Υ is conditionally compact in the X-topology.

It would be desirable to replace conditions (1) and (2) by similar conditions requiring only quasi-uniform convergence on Y and X rather than on the full sets $\mathfrak{Y}$ and $\mathfrak{X}$. Unfortunately, we have been unable to do this in the general case. However, in a situation that is of some importance for later applications, namely when $\mathfrak{X}=\lambda(X)$ this replacement is possible. This occurs when X is a compact space and $\{\eta_y;\, y\in Y\}$ is a collection of continuous functions on X.

3.6. Lemma. *If $\mathfrak{X}=\lambda(X)$, then every U-net which converges quasi-uniformly on Y also converges quasi-uniformly on $\mathfrak{Y}$.*

Proof. Let (x_α) be a U-net which converges to $\mathfrak{x}_0=\lambda(x_0)$ quasi-uniformly on Y. Then given $\epsilon>0$, α_0 there exist $\alpha_1, \cdots, \alpha_r \geqq \alpha_0$ such that for each $y\in Y$,

$$\min_{1 \leqq i \leqq r} | h(x_{\alpha_i}, y) - h(x_0, y) | < \epsilon.$$

We shall show that the same indices are effective for $\mathfrak{Y}$, when ϵ is replaced by 3ϵ. The set $N(\mathfrak{y}; x_0, \epsilon)$ is open in the X-topology of $\mathfrak{Y}$. Since $\mu(Y)$ is dense in this topology, there exists some $y_0 \in Y$ such that

$$\mu(y_0) \in N(\mathfrak{y}; x_0, \epsilon) \cap \bigcap_{i=1}^{r} N(\mathfrak{y}; x_{\alpha_i}, \epsilon).$$

Let α_i be one of the indices $\alpha_1, \cdots, \alpha_r$ for which

$$| h(x_{\alpha_i}, y_0) - h(x_0, y_0) | < \epsilon.$$

Then, we have

$$| h_2(x_{\alpha_i}, \mathfrak{y}) - h_2(x_0, \mathfrak{y}) | \leqq | h_2(x_{\alpha_i}, \mathfrak{y}) - h(x_{\alpha_i}, y_0) | + | h(x_{\alpha_i}, y_0) - h(x_0, y_0) | + | h(x_0, y_0) - h_2(x_0, \mathfrak{y}) | < 3\epsilon,$$

which proves the assertion.

3.7. THEOREM. *If $\mathfrak{X} = \lambda(X)$, then X and Y are weakly paired by h if and only if every U-net in X converges quasi-uniformly on Y.*

We now pass to the study of a stronger type of pairing of two sets where quasi-uniform convergence is replaced by uniform convergence. In view of Theorem 2.9, the uniform convergence on a dense set implies uniform convergence on the entire space. This fact will render the present analysis much more manageable. The next theorem is essentially due to Kakutani [12].

3.8. THEOREM. *The following statements are equivalent:*

(1) *Every U-net in X converges uniformly on Y;*

(2) *Every U-net in Y converges uniformly on X;*

(3) *Given $\epsilon > 0$, there is a partition $X = \bigcup_{i=1}^{n} A_i$ such that if x', x'' belong to the same A_i, then $|h(x', y) - h(x'', y)| < \epsilon$, $y \in Y$;*

(4) *Given $\epsilon > 0$, there is a partition $Y = \bigcup_{j=1}^{m} B_j$ such that if y', y'' belong to the same B_j, then $|h(x, y') - h(x, y'')| < \epsilon$, $x \in X$.*

Proof. Clearly (3) implies (1), for every U-net is ultimately in some one of the A_i. We will show, by contradiction, that (2) implies (3). For if (3) is not true, there exists an $\epsilon_0 > 0$ such that for any partition π of X there is a $y_\pi \in Y$ and two points x'_π, x''_π in the same set of the partition π such that

$$[*] \qquad | h(x'_\pi, y_\pi) - h(x''_\pi, y_\pi) | \geqq \epsilon_0.$$

By Theorem 1.5 we may suppose that (y_π) is a U-net in Y and thus converges uniformly on X to an element $\mathfrak{y} \in \mathfrak{Y}$. By Theorem 2.9, we see that the functions (η_{y_π}) converge to $\eta_{\mathfrak{y}}$ uniformly on $\mathfrak{X}$. Since $\eta_{\mathfrak{y}}$ is a continuous function on the compact Hausdorff space $\mathfrak{X}$, there exists a finite number of nonvoid sets $\mathfrak{A}_i = \{\mathfrak{x}; |\eta_{\mathfrak{y}}(\mathfrak{x}) - \eta_{\mathfrak{y}}(\mathfrak{x}_i)| < \epsilon_0/4\}$, $i = 1, \cdots, n$, covering $\mathfrak{X}$. Select a partition

π of X which refines $\{\lambda^{-1}(\mathfrak{A}_i)\}$ and which is such that $|\eta_{y_\pi}(x)-\eta_{\mathfrak{y}}(x)|<\epsilon_0/4$, $x\in X$, which is possible by (2). Then

$$|h_2(x, \mathfrak{y}) - h(x, y_\pi)| < \epsilon_0/4, \qquad x \in X,$$

and if x', x'' belong to the same subset of π,

$$|h_2(x', \mathfrak{y}) - h_2(x'', \mathfrak{y})| < \epsilon_0/4.$$

Therefore,

$$\begin{aligned} |h(x_\pi', y_\pi) - h(x_\pi'', y_\pi)| \leqq{} & |h(x_\pi', y_\pi) - h_2(x_\pi', \mathfrak{y})| \\ & + |h_2(x_\pi', \mathfrak{y}) - h_2(x_\pi'', \mathfrak{y})| \\ & + |h_2(x_\pi'', \mathfrak{y}) - h(x_\pi'', y_\pi)| < \epsilon_0, \end{aligned}$$

but this contradicts [*].

3.9. Definition. If X and Y are two sets which are paired by the function h and satisfy the conditions of Theorem 3.8, we say that X and Y *are strongly paired by h*.

The interest in strongly paired sets is that in this case Ξ is conditionally compact in the topology of uniform convergence on Y (or on $\mathfrak{Y}$), and a similar statement holds for Υ.

4. **Examples and applications.** This section gives some simple examples of strong and weak pairing of subsets of Banach spaces. As immediate corollaries of these notions we derive well-known theorems of Mazur [14], Schauder [18], and Gantmacher [8].

Let E be a Banach space and E^* its adjoint. We denote by S and S^* the solid closed unit spheres of E and E^*. It is natural to inquire what subsets of E [resp. E^*] can be weakly or strongly paired with S^* [resp. S] under the natural pairing $h(x, x^*)=x^*(x)$. If Γ is a collection of functions on a vector space X, the Γ-topology of X is the topology whose subbasic neighborhoods are given by $N(x_0; \gamma, \epsilon)=\{x\in X; |\gamma(x)-\gamma(x_0)|<\epsilon\}$. Thus the E^*-topology of E is its weak topology, and the E-topology of E^* is its weak* topology.

4.1. Theorem. *Let $h(x, x^*)=x^*(x)$ for $x\in E$, $x^*\in E^*$. Then*:

(1) *The set $A\subseteq E$ can be paired with S^* by the function h if and only if it is bounded*;

(2) *A can be weakly paired with S^* if and only if it is conditionally weakly compact*;

(3) *A can be strongly paired with S^* if and only if it is conditionally* (*strongly*) *compact*.

Proof. The first statement follows immediately from the uniform boundedness theorem. Suppose that A is conditionally weakly compact, then the weak closure $\overline{A}$ is bounded and weakly compact. Further the embedding λ: $A\rightarrow\mathfrak{A}$ is one-to-one, by the Hahn-Banach theorem, and it is clear that $\overline{A}$

is homeomorphic with $\mathfrak{A}$. Now S^* is compact in the E-topology and hence in the A-topology; thus $\mathfrak{S}^* = S^*$. It is obvious that each point in $\mathfrak{A}$ is a continuous function on S^*, so that A and S^* are weakly paired. Conversely, if A is weakly paired with S^*, then A is conditionally S^*-compact, hence conditionally weakly compact.

If A is strongly paired with S^*, then for $\epsilon > 0$ there is a decomposition $A = \bigcup_{i=1}^n A_i$ where $x, x' \in A_i$ implies $|x^*(x-x')| < \epsilon$ for all $x^* \in S^*$. Hence $\|x - x'\| < \epsilon$. Let $x_i \in A_i$, then the spheres $S(x_i, \epsilon)$, $i = 1, \cdots, n$, cover A so that A is totally bounded. The sufficiency of this part follows similarly.

Before investigating the weak pairing of S and a set $B \subseteq E^*$, we make the following observation: If $B \subseteq E^*$ is compact in the E-topology, then B is compact in the E^{**}-topology if and only if the restriction to B of every linear functional in S^{**} is continuous in the E-topology of B.

The necessity of the condition is manifest. The condition implies that the E^{**}-topology of B is weaker than the E-topology; thus B is E^{**}-compact.

4.2. Theorem. *Let* $h(x, x^*) = x^*(x)$ *for* $x \in E$, $x^* \in E^*$. *Then*:

(1) *S can be paired with a set $B \subseteq E^*$ by the function h if and only if B is bounded*;

(2) *S can be weakly paired with B if and only if B is conditionally weakly* (E^{**}-) *compact*;

(3) *S can be strongly paired with B if and only if B is conditionally* (*strongly*) *compact.*

Proof. Statements (1) and (3) follow readily. If S and B are paired, the mapping $\mu: B \to \mathfrak{B}$ is one-to-one. It is readily seen that $\mathfrak{B}$ is homeomorphic with $\overline{B}$, the E-closure of B. By Goldstine's [10] theorem, S is dense in the E^*-topology of S^{**} and thus $\mathfrak{S}$ is homeomorphic with S^{**} in the B-topology. Now if S and B are weakly paired, every point in $\mathfrak{S} = S^{**}$ gives a continuous function on the S-compact set $\overline{B}$. Thus $\overline{B}$ is E^{**}-compact, and B is conditionally E^{**}-compact. Conversely, if B is conditionally E^{**}-compact, then its E^{**}-closure coincides with its E-closure, $\overline{B}$. Since they are both Hausdorff, the E- and E^{**}-topologies coincide on $\overline{B}$, so that every point in $S^{**} = \mathfrak{S}$ gives a continuous function on $\overline{B} = \mathfrak{B}$. Thus S and B are weakly paired by h.

4.3. Definition. Let T be a function, not assumed to be continuous or linear, which maps a Banach space E into a Banach space E_1. Let $S = \{x; x \in E, \|x\| \leqq 1\}$ and $S_1 = \{y; y \in E_1, \|y\| \leqq 1\}$. We say that T is *strongly* [*weakly*] *compact* if $T(S)$ is conditionally strongly [weakly] compact. We say that T has an *adjoint* if there exists a function $T^*: E_1^* \to E^*$ for which

$$h_T(x, y^*) = y^*(Tx) = (T^*y^*)x, \qquad x \in E,\ y^* \in E_1^*.$$

This definition clearly coincides with the usual notions of a [weakly] compact mapping and adjoint when T is a linear operator.

4.4. THEOREM. *Suppose that $T:E\to E_1$ has an adjoint T^*. Then T is strongly [weakly] compact if and only if T^* is strongly [weakly] compact.*

Proof. The proof follows from the equivalence of the following statements:

(1) $T(S)$ is conditionally strongly [weakly] compact.

(2) $T(S)$ is strongly [weakly] paired with S_1^* by the function $h_1(y, y^*) = y^*(y)$.

(3) S and S_1^* are strongly [weakly] paired by the function h_T.

(4) S is strongly [weakly] paired with $T^*(S_1^*)$ by the function $h(x, x^*) = x^*(x)$.

(5) $T^*(S^*)$ is conditionally strongly [weakly] compact.

As another application of Theorem 3.8, we derive a well-known theorem of Mazur [14].

4.5. THEOREM. *A subset of a Banach space is conditionally strongly compact if and only if the closed convex set generated by it is strongly compact.*

Proof. The necessity is trivial. If the set $A\subseteq E$ is conditionally compact it can be strongly paired with S^* by $h(x, x^*)=x^*(x)$. By Theorem 3.8, given $\epsilon>0$, there is a partition $S^*=\bigcup_{j=1}^m B_j$ such that if x_1^* and x_2^* belong to the same B_j, then

$$\left| (x_1^* - x_2^*)x \right| \leqq \epsilon$$

for all $x\in A$. The same inequality remains valid for all x in the set K consisting of limits of convex combinations of elements of A. Thus K can be strongly paired with S^*, and is conditionally compact. Since E is complete, the closed set K is strongly compact.

5. **Applications to $C(X)$.** We shall now apply the abstract results of §3 to obtain information concerning continuous functions on a compact topological space.

5.1. DEFINITION. If X is a set and F a collection of scalar-valued functions on X, then the *topology of pointwise convergence* is the topology on F generated by the subbasic sets $N(f_0; x, \epsilon) = \{f\in F; |f(x)-f_0(x)| < \epsilon\}$.

Thus this topology is identical with the relative product topology on F; it is clear that $f_\alpha\to f$ in this topology if and only if $f_\alpha(x)\to f(x)$ for all $x\in X$.

If F is a set of bounded functions on some set X, then X and F are paired in the sense of Definition 3.1 by the function h, defined on $X\times F$ by $h(x, f) = f(x)$, provided only that for each $x\in X$ there exists an $M_x>0$ such that $|f(x)| < M_x$, $f\in F$. That is, X and F are paired if and only if F is *pointwise bounded* on X.

Now suppose that X is a compact space and that F consists of continuous functions on X. Then since the F-topology on X is weaker than the original topology, it follows that X is F-compact and $\lambda(X)=\mathfrak{X}$. In addition, the mapping $\mu: F\to \mathrm{P}I_x$ is one-to-one; consequently, μ is a homeomorphism of F into a subspace of $\mathfrak{F}$, where F has the topology of pointwise convergence and $\mathfrak{F}$ has the X-topology.

5.2. DEFINITION. A collection F of functions on a set X is said to be *quasi-equicontinuous* on X if every U-net in X converges quasi-uniformly on F.

We emphasize that this notion is independent of the topology (if any) of the set X. It is worth observing, however, that if X is a topological space and F a quasi-equicontinuous family of functions on X, then each $f \in F$ is continuous. In a general space we shall require the use of U-nets, but in a compact space the following criterion is available:

5.3. LEMMA. *A collection F of (continuous) functions on a compact space X is quasi-equicontinuous on X if and only if $x_\alpha \to x_0$ in X implies that $f(x_\alpha) \to f(x_0)$ quasi-uniformly on F.*

Proof. If X is compact, every U-net converges to some point(4) in X, so the sufficiency is immediate. Conversely, let F be quasi-equicontinuous and suppose $x_\alpha \to x_0$ in X. By Theorem 1.5, there is a subnet of (x_α) which is a U-net and thus converges quasi-uniformly. The necessity now follows from Lemma 2.4.

With these notions, we can now prove one of our main results (compare [5; 20]).

5.4. THEOREM. *Let X be a compact topological space. Then a collection F of continuous functions on X is conditionally compact in the topology of pointwise convergence if and only if F is pointwise bounded and quasi-equicontinuous on X.*

Proof. Since it is clear that pointwise boundedness is a necessary condition, we suppose that F enjoys this property. Then X and F are paired by $h(x, f) = f(x)$. In view of Theorems 3.4 and 3.7 the following statements are equivalent: F is quasi-equicontinuous on X; X and F are weakly paired; $\mathfrak{F}$ consists of continuous functions on $\mathfrak{X} = \lambda(X)$. Since $\mathfrak{F}$ is compact in the X-topology, the theorem follows.

The relation between compactness in the sense of the Heine-Borel theorem and sequential compactness is often complicated. It follows from a result due to A. Grothendieck [11] that these notions coincide in the case we are studying. We now present a theorem closely related to some of Grothendieck's results.

5.5. THEOREM. *Let X be a compact topological space, let $F \subseteq C(X)$, and let D be a dense subset of X. Then the following statements are equivalent:*

(1) *F is conditionally compact in the topology of pointwise convergence;*

(2) *F is pointwise bounded and quasi-equicontinuous on X;*

(3) *F is pointwise bounded and if F_0 is a denumerable subset of F, if $x_0 \in X$*

(4) See Kelley [13].

and (x_n) *is a sequence in* D *for which* $f(x_n) \to f(x_0)$, $f \in F_0$, *then the convergence is quasi-uniform on* F_0;

(4) *From every sequence in* F *one can extract a subsequence which converges at each point of* X *to a continuous limit.*

Proof. We have already seen that (1) and (2) are equivalent. Suppose that (2) and the hypotheses of (3) are true. Then there is a U-net (y_α) which is a subnet of the sequence (x_n). Let $y_0 = \lim y_\alpha$, then $f(y_\alpha) \to f(y_0)$ for all $f \in F$ and *a fortiori* for $f \in F_0$. It is readily seen that $f(y_0) = f(x_0)$ for $f \in F_0$. Since $f(y_\alpha) \to f(y_0)$ quasi-uniformly on F_0, the conclusion of (3) follows from Lemma 2.4.

We now show that (3) implies (4). Let $F_0 = (f_i)$ be a sequence of functions in F and set

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}, \qquad x, y \in X.$$

We define an equivalence relation $x \sim y$ if and only if $\rho(x, y) = 0$, or if and only if $f_i(x) = f_i(y)$, $i = 1, 2, \cdots$. Denote the equivalence classes by corresponding Greek letters, and let X_ρ be the collection of these classes. It is clear that X_ρ is a metric space under the metric ρ. Since the natural mapping $J: X \to X_\rho$ is continuous, X_ρ is a compact metric space and therefore separable. Further since D is dense in X it is clear that $J(D)$ contains a countable set Δ_0 which is dense in X_ρ. For $f_i \in F_0$, we define ϕ_i on X_ρ by $\phi_i(\xi) = f_i(x)$ where $x \in \xi$. It follows from the definition of the equivalence relation and ρ that ϕ_i is well-defined and continuous.

By a diagonal argument, we can pick a subsequence (ψ_n) of (ϕ_n) which converges at each point of the countable set Δ_0. Define $\psi_0(\zeta) = \lim_{n \to \infty} \psi_n(\zeta)$ for $\zeta \in \Delta_0$. We shall show that (ψ_n) converges at every point of X_ρ and that the limit function is continuous.

Let ξ_0 be an arbitrary point of X_ρ. We first show that if (π_i) is any subsequence of (ψ_n) such that $\pi_i(\xi_0) \to L$ and if $(\zeta_r) \subseteq \Delta_0$ with $\zeta_r \to \xi_0$, then $\psi_0(\zeta_r) \to L$. To prove this it is sufficient to show that given any subsequence (η_k) of (ζ_r), a positive number ϵ and an integer K, there is a $k \geqq K$ such that

$$[*] \qquad |\psi_0(\eta_k) - L| < \epsilon.$$

Since $\eta_k \to \xi_0$, $\lim_{k \to \infty} \pi_i(\eta_k) = \pi_i(\xi_0)$ for each i. From (3) it follows that there exists $k_1, \cdots, k_p \geqq K$ such that for any i there is an integer j with $1 \leqq j \leqq p$ such that

$$|\pi_i(\eta_{k_j}) - \pi_i(\xi_0)| < \epsilon/3.$$

Now (π_i) converges at ξ_0 and (π_i) is a subsequence of (ψ_n), which converges on Δ_0. Thus there is an index i such that

$$|\pi_i(\xi_0) - L| < \epsilon/3,$$
$$|\pi_i(\eta_{k_j}) - \psi_0(\eta_{k_j})| < \epsilon/3, \qquad j = 1, \cdots, p.$$

Since $k_j \geqq K$, these inequalities prove inequality [*].

We now assert that $\lim_{n\to\infty} \psi_n(\xi)$ exists for each $\xi \in X_\rho$. For, if this is not true there exists a ξ_0, two numbers L, L' with $L \neq L'$ and two subsequences (π_i) and (π_i') of (ψ_n) such that $\pi_i(\xi_0) \to L$ and $\pi_i'(\xi_0) \to L'$. But if $(\zeta_r) \subseteq \Delta_0$ and $\zeta_r \to \xi_0$, the argument just completed shows that $L = \lim \psi_0(\zeta_r) = L'$, which is a contradiction.

Define ψ_0 on X_ρ by $\psi_0(\xi) = \lim_{n\to\infty} \psi_n(\xi)$. It must be shown that ψ_0 is continuous on X_ρ. Let $\xi_n \to \xi_0$; since Δ_0 is dense in X_ρ, there are sequences $(\zeta_{n,i}) \subseteq \Delta_0$ with $\zeta_{n,i} \to \xi_n$ for $n = 0, 1, 2, \cdots$. The argument above implies that $\psi_0(\zeta_{n,i}) \to \psi_0(\xi_n)$ for $n = 0, 1, 2, \cdots$. Further, the diagonal sequence $(\zeta_{n,n})$ converges to ξ_0 and hence $\psi_0(\zeta_{n,n}) \to \psi_0(\xi_0)$. It follows readily that $\psi_0(\xi_n) \to \psi_0(\xi_0)$ so that ψ_0 is continuous on X_ρ.

Define g_0 on X by $g_0(x) = \psi_0(J(x))$, so that g_0 is continuous. Let (g_n) be the subsequence of (f_n) which corresponds to the subsequence (ψ_n) of (ϕ_n). Then it is clear that $g_n(x) \to g_0(x)$ for all $x \in X$. Hence F is sequentially compact and (4) is proved.

That (4) implies (1) is a consequence of a theorem of Grothendieck [11, p. 171].

For the sake of completeness we cite the corresponding result for compactness in the topology of uniform convergence on X; i.e., in the norm of the Banach space $C(X)$.

5.6. Definition. A collection F of functions on a set X is said to be *equicontinuous on* X if every U-net in X converges uniformly on F.

We leave it to the reader to show that if X is compact, Definition 5.6 coincides with the usual notion of equicontinuity: given any positive number ϵ and $x \in X$ there exists a neighborhood $N(x)$ such that if $y \in N(x)$ and $f \in F$ then $|f(y) - f(x)| < \epsilon$.

5.7. Theorem. *Let X be a compact topological space, let F be a bounded set in $C(X)$, and let D be dense in X. Then the following statements are equivalent:*

(1) *F is conditionally compact in the topology of uniform convergence;*

(2) *Given $\epsilon > 0$ there is a partition $X = \bigcup_{i=1}^n A_i$ such that if x and y belong to the same A_i, then*

$$|f(x) - f(y)| < \epsilon, \qquad f \in F;$$

(3) *F is equicontinuous on X;*

(4) *If F_0 is a denumerable subset of F, $x_0 \in X$, and (x_n) is a sequence in D for which $f(x_n) \to f(x_0)$, $f \in F_0$, then the convergence is uniform on F_0;*

(5) *From every sequence in F one can extract a subsequence which converges uniformly on X.*

Proof. Since $C(X)$ is a Banach space in this topology, (1) and (5) are equivalent to each other and to the total boundedness of F. By Theorem 3.8,

(2) and (3) are also equivalent to (1). It remains to prove that (4) and (5) are equivalent.

If (4) is true, then by the proof of the preceding theorem, every sequence has a subsequence which converges on X to a continuous function. We shall show that if $(f_n)\subseteq F$ and $f_n(x)\rightarrow f_0(x)$, $x\in X$, for some continuous f_0, then the convergence is uniform. If this were not so, there would be an $\epsilon>0$, a sequence $(x_k)\subseteq X$, and a subsequence (f_{n_k}) of (f_n) such that

$$[*] \qquad |f_{n_k}(x_k) - f_0(x_k)| \geqq \epsilon, \qquad k = 1, 2, \cdots.$$

By dropping to a subsequence of (x_k) and renumbering, we may suppose that $J(x_k)\rightarrow J(x_0)$ in X_ρ (the notation is that of the preceding theorem). Since $f(x_k)\rightarrow f(x_0)$, $f\in F_0$, condition (4) and the continuity of f_0 imply that there exists an integer I such that if $i\geqq I$ then

$$|f_{n_k}(x_i) - f_{n_k}(x_0)| < \epsilon/3, \qquad k = 1, 2, \cdots,$$
$$|f_0(x_i) - f_0(x_0)| < \epsilon/3.$$

Since $f_{n_k}(x_0)\rightarrow f_0(x_0)$, it follows that if $k\geqq K$ and $i\geqq I$ then

$$\begin{aligned} |f_{n_k}(x_i) - f_0(x_i)| &\leqq |f_{n_k}(x_i) - f_{n_k}(x_0)| + |f_{n_k}(x_0) - f_0(x_0)| \\ &\quad + |f_0(x_0) - f_0(x_i)| < \epsilon, \end{aligned}$$

which contradicts [*]. Hence (4) implies (5); the proof that (5) implies (4) is similar and will be omitted.

5.8. Remark. In Theorem 5.4 the functions may have their range in a Hausdorff uniform space; in Theorems 5.5 and 5.7 the proof requires the range space to be metric. In all three it is assumed that for each $x\in X$, the set $\{f(x); f\in F\}$ is conditionally compact.

6. **Weak compactness in $C(X)$.** Throughout this section we suppose that X is a compact Hausdorff space. Theorem 5.7 gives reasonably complete information concerning the conditionally (strongly) compact subsets of the Banach space $C(X)$. The weakly compact subsets of this space can be treated in essentially the same manner as the sets which are conditionally compact in the topology of pointwise convergence. That this is possible is indicated by a theorem of Grothendieck [11, p. 182], but we will find it convenient to derive the conditions directly, relying on the more familiar theorem of Eberlein [7].

6.1. Theorem. *Let X be a compact Hausdorff space, let $F\subseteq C(X)$ and let D be a dense subset of X. Then the following statements are equivalent:*

(1) *F is conditionally compact in the weak topology of $C(X)$;*

(2) *F is bounded and conditionally compact in the topology of pointwise convergence;*

(3) *F is bounded and quasi-equicontinuous on X;*

(4) *F is bounded and if F_0 is a denumerable subset of F, if $x_0 \in X$ and (x_n) is a sequence in D for which $f(x_n) \to f(x_0)$, $f \in F_0$, then the convergence is quasi-uniform on F_0;*

(5) *F is weakly sequentially compact.*

Proof. It is well known that (1) implies that F is bounded; further the weak topology of $C(X)$ is stronger than the pointwise topology so that (2) follows. That (2) implies (3) and (3) implies (4) were proved in Theorem 5.5. The implication of (5) follows from Theorem 5.5 and the Lebesgue dominated convergence theorem. Finally, if (5) is valid, a theorem of Phillips [17] implies that the weak closure $\bar{F}$ of F is weakly sequentially compact and Eberlein's theorem asserts that $\bar{F}$ is weakly compact. Thus (5) implies (1) and the cycle is complete.

7. **Arbitrary topological spaces.** In this section we will be concerned with two Banach spaces $BC(Q)$ and $M(S)$ and show that the preceding remarks on $C(X)$, X compact Hausdorff, can be applied to them.

If Q is an arbitrary topological space, we denote the collection of all bounded continuous real- or complex-valued functions by $BC(Q)$. If S is an abstract set, we denote the collection of all bounded real- or complex-valued functions by $M(S)$. Both of these spaces are Banach spaces under the supremum norm, and $BC(Q)$ is a closed linear manifold in $M(Q)$. On the other hand, $M(S)$ may be regarded as $BC(S)$ by equipping S with its discrete topology.

Let βQ be the Stone-Čech compactification (see [22]) of Q. Then, in a natural sense, the space Q can be mapped (possibly in a many-to-one manner) on a dense subset of the compact Hausdorff space βQ and the mapping of Q onto its range is continuous and open. Further $BC(Q)$ and $C(\beta Q)$ are isometrically isomorphic.

It is a consequence of the Lebesgue theorem of dominated convergence that if Q is compact Hausdorff, then a sequence (f_n) in $C(Q)$ converges weakly to $f_0 \in C(Q)$ if and only if it is bounded and converges pointwise on Q. If the space Q is not assumed to be compact then the measures representing the continuous linear functionals may not be countably additive so the Lebesgue theorem does not apply. The next theorem enables us to derive a condition for weak convergence in $BC(Q)$.

7.1. Theorem. *Let A be a dense subset of a compact topological space X, and suppose that a sequence (f_n) of continuous functions converges at every point of A to a continuous limit f_0. Then (f_n) converges to f_0 at every point of X if and only if every subsequence of (f_n) converges to f_0 quasi-uniformly on A.*

Proof. The theorem of Arzelà implies that the condition is necessary. To prove the sufficiency, suppose that $f_n(x_0)$ does not converge to $f_0(x_0)$. Then there exists an ϵ and a subsequence (g_k) of (f_n) such that $|g_k(x_0) - f_0(x_0)| > \epsilon$, for $k=1, 2, \cdots$. Let $k_1, \cdots, k_r$ be the indices corresponding to ϵ and $k=1$ guaranteed by the quasi-uniform convergence of (g_k). Then $U_i=\{x; |g_{k_i}(x) -f_0(x)| >\epsilon\}$ is an open set containing x_0 for $i=1, \cdots, r$. Since A is dense in X there exists a point $a\in A\cap U_1\cap \cdots \cap U_r$. For this point a we have

$$|g_{k_i}(a) - f_0(a)| > \epsilon, \qquad i = 1, \cdots, r,$$

but this contradicts the quasi-uniform convergence of (g_k) on A.

The same proof can be used to prove

7.2. COROLLARY. *Let (f_α) be a net of continuous functions on a compact space X which converges on a dense set A to a continuous function f_0. Then (f_α) converges to f_0 on all of X if and only if every subnet of (f_α) converges quasi-uniformly on A.*

Applying the theorem to a sequence in $BC(Q)$ we have:

7.3. COROLLARY. *A sequence (f_n) in $BC(Q)$ converges weakly to $f_0\in BC(Q)$ if and only if it is bounded and every subsequence of (f_n) converges quasi-uniformly on Q.*

This corollary was proved by Sirvint [20] who demonstrated that the convergence condition is equivalent to a condition given by Banach [4, p. 219].

We recall that the notions of equicontinuity and quasi-equicontinuity of a family of functions as defined in §5 are independent of the topology of the underlying space. Using the fact that the space Q is a dense subset of its Stone-Čech compactification βQ and that $BC(Q)$ is isometrically isomorphic with $C(\beta Q)$ we derive the next two theorems as trivial modifications of the corresponding theorems for $C(\beta Q)$.

7.4. THEOREM. *The following statements are equivalent for a bounded subset $F\subseteq BC(Q)$:*

(1) *F is conditionally (strongly) compact;*

(2) *F is equicontinuous on Q;*

(3) *If F_0 is a denumerable subset of F and (q_n) is a sequence in Q for which $(f(q_n))$ converges for each $f\in F_0$, then the convergence is uniform on F_0;*

(4) *For any positive ϵ there is a partition $Q=\bigcup_{i=1}^{n} A_i$ such that if q', q'' belong to the same A_i then $|f(q')-f(q'')| <\epsilon, f\in F$.*

7.5. THEOREM. *The following statements are equivalent for a bounded subset $F\subseteq BC(Q)$:*

(1) *F is conditionally weakly compact;*

(2) *F is quasi-equicontinuous on Q;*

(3) *If F_0 is a denumerable subset of F and (q_n) is a sequence in Q for which $(f(q_n))$ converges for each $f\in F_0$, then the convergence is quasi-uniform on F_0.*

These conditions also apply to $M(S)$ without change.

8. **Analytic and almost-periodic functions.** Let D be a bounded open set in the complex plane and $\overline{D}$ be its closure. By $A(D)$ we signify the collection of all functions which are analytic in D and continuous on $\overline{D}$. With the norm $\|f\| = \sup_{z \in D} |f(z)|$, this set is a Banach space, and it is clear that $A(D)$ is a closed linear manifold of the space $C(\overline{D})$.

It follows readily from Theorem 5.7 that a set $F \subseteq A(D)$ is conditionally compact in the norm topology if and only if it is bounded and equicontinuous on $\overline{D}$. This criterion makes no use of the fact that the functions are analytic and hence that boundedness on $\overline{D}$ implies equicontinuity on any closed set contained in D. More essential use of this fact is utilized in

THEOREM 8.1. *A set $F \subseteq A(D)$ is weakly compact if and only if F is bounded and quasi-equicontinuous on the boundary of D. If these conditions hold, then from any sequence in F one can extract a subsequence which converges to a limit in $A(D)$ and the convergence is uniform on every closed subset of D.*

This follows readily from Theorem 6.1 and the well-known convergence theorem of Vitali.

Professor S. Kakutani has pointed out that the preceding work renders simple proofs of the equivalence of left and right almost-periodicity. In the case of weakly almost periodic functions (as defined by Eberlein [6]) this was proved by Grothendieck [11]. We give a short symmetric proof here.

Let S be a semi-group and let f be a bounded scalar-valued function on S. For each $a \in S$, define the left [resp. right] a-translate of f by $f^a(x) = f(ax)$ [resp. $f_a(x) = f(xa)$], $x \in S$. We say that f is strongly [weakly] left almost-periodic if $\{f^a; a \in S\}$ is a conditionally strongly [weakly] compact subset of $M(S)$. The definitions of strong and weak right almost-periodicity are exactly similar.

8.2. THEOREM. *A bounded function on a semi-group is strongly [weakly] left almost-periodic if and only if it is strongly [weakly] right almost-periodic.*

Proof. Let $S_1 = S_2 = S$ and let h be defined on $S_1 \times S_2$ by $h(x, y) = f(xy)$, $x, y \in S$. Then the strong case follows immediately from Theorems 7.4 and 3.8. If f is weakly left almost periodic, then the weak closure $\overline{S}_1$ of $\{f^a; a \in S\}$ is weakly compact and hence compact in the topology of pointwise convergence on S. It is evident that $\mathfrak{S}_1$ is homeomorphic with $\overline{S}_1$ in this topology. From Theorems 7.5 and 3.4, it follows that S_1 and S_2 are weakly paired by h. Another application of Theorem 7.5 proves that the set $\{f_a; a \in S\}$ is weakly compact. Hence f is weakly right almost periodic.

9. **Compactness in Banach spaces.** As easy corollaries of the work of the preceding sections, we cite some results which are known for separable Banach spaces. Throughout this section E denotes a Banach space, E^* its adjoint, S and S^* the respective closed solid unit spheres. In the case of a separable space, the strong part of the next theorem was proved by Gelfand [9] and

the weak part by Sirvint [20]([5]).

9.1. THEOREM. *A subset* $A \subseteq E$ *is conditionally [weakly] compact if and only if it is bounded and for every bounded net* (x_α^*) *in* E^* *with* $x_\alpha^*(x) \rightarrow x_0^*(x)$, $x \in E$, *the convergence is [quasi-] uniform on* A.

Proof. This follows immediately from Theorem 4.1 and Theorem [3.7] 3.8. It is frequently useful to observe that the necessity remains true even if the net is not assumed to be bounded.

9.2. THEOREM. *The following statements are equivalent*:

(1) *The net* $(x_\alpha) \subseteq E$ *converges [weakly] to* x_0;

(2) *Every subnet of* (x_α) *converges [weakly] to* x_0;

(3) *The net* (x_α) *converges to* x_0 *[quasi-] uniformly on* S^*;

(4) *Every subnet of* (x_α) *converges to* x_0 *[quasi-] uniformly on* S^*.

The proof of this theorem will be omitted. We observe, however, that it implies that if (x_α^*) is a net in E^* which converges to x_0^* in the weak topology of E^* (i.e., the E^{**}-topology of E^*) then every subnet of (x_α^*) converges quasi-uniformly on S^{**} and hence on S. It is not without interest that the weakly convergent nets in E^* can be characterized by their behavior on S.

9.3. THEOREM. *A net* $(x_\alpha^*) \subseteq E^*$ *converges weakly to* x_0^* *in the* E^{**}*-topology if and only if every subnet of* (x_α^*) *converges to* x_0^* *quasi-uniformly on* S.

Proof. The necessity follows from Theorem 9.2 as already observed. The sufficiency follows from Goldstine's theorem [10] by an argument which is exactly as in the proof of Lemma 3.6. Consequently, we shall omit it. We observe that this theorem implies the well-known condition that E is reflexive if and only if S is E^*-compact.

Using these facts we can prove readily some very convenient characterizations of weakly compact linear operators which are due to Gantmacher [8]—see also Nakamura [15].

9.4. THEOREM. *Let* T *be a bounded linear operator from a Banach space* E *to a Banach space* E_1. *Then the following are equivalent*:

(1) T *is weakly compact*;

(2) T^* *is continuous with the* E_1*-topology on* E_1^* *and the* E^{**}*-topology on* E^*;

(3) $T^{**}(E^{**})$ *is a subset of the natural embedding of* E_1 *into* E_1^{**}.

Proof. Let (y_α^*) be an arbitrary net in E_1^* which converges to y_0^* in the E_1-topology of E_1^*, and let x^{**} be an arbitrary elements of E^{**}. We shall show that each of the following statements implies its successor. $T(S)$ is conditionally compact in the E_1^*-topology of E_1. As remarked after Theorem 9.1,

([5]) It should be mentioned that in the separable case one can replace nets by sequences of functionals. Phillips [16] proved that Gelfand's use of sequences in the nonseparable case is incorrect, but it is possible that some use of sequences may suffice.

every subnet of (y_α^*) converges to y_0^* quasi-uniformly on $T(S)$. Every subnet of $(T^*y_\alpha^*)$ converges to $T^*y_0^*$ quasi-uniformly on S. By Theorem 9.3, $(T^*y_\alpha^*)$ converges to $T^*y_0^*$ in the E^{**}-topology of E^*. T^* is continuous with the E_1-topology on E_1^* and the E^{**}-topology on E^*. The functional $T^{**}x^{**} = x^{**}T^*$ is continuous with the E_1-topology on E_1^*. $T^{**}(E^{**})$ is a subset of the natural embedding of E_1 into E_1^{**}. That (3) implies (1) follows from standard arguments.

9.5. Remark. Using similar techniques one can show that a bounded linear operator $T: E \to E_1$ is compact if and only if T^* maps bounded nets which converge in the E_1-topology of E_1^* into nets which converge in the norm of E^*.

10. **Operators with range in $BC(Q)$.** We now extend an elementary result concerning the representation of [weakly] compact linear operators from an arbitrary Banach space E into the Banach space $BC(Q)$. For $Q = [0, 1]$, this representation was given by Gelfand [9] in the compact case and Sirvint [20] in the weakly compact case.

The following lemma follows easily from the definitions and well-known results:

10.1. Lemma. *If Q is a topological space, then each point $q \in Q$ gives rise to a continuous linear functional y_q^* on $BC(Q)$ defined by $y_q^*(f) = f(q)$, $f \in BC(Q)$. The Stone-Čech compactification βQ of Q is homeomorphic with the closure of such functionals in the $BC(Q)$-topology of the unit sphere of $BC^*(Q)$.*

Let T be an arbitrary bounded linear operator mapping a Banach space E into $BC(Q)$; then T^* is continuous with the $BC(Q)$-topology on $BC^*(Q)$ and the E-topology on E^*. By the preceding lemma we can regard βQ as a compact subset of $BC^*(Q)$, and with obvious identifications we visualize Q as a dense subset of βQ. Let τ be the restriction of T^* to Q. Thus $\tau: Q \to E^*$ is continuous with the E-topology on E^* and we have

$$[*] \qquad [T(x)](q) = [\tau(q)](x), \qquad x \in E,\ q \in Q.$$

It is readily seen that $\|T\| = \sup \|\tau(q)\|$, the supremum being taken over Q. Conversely, if τ is a map of Q into a bounded portion of E^* which is continuous with E^* in its E-topology, then the function T defined by [*] is a linear operator from E to $BC(Q)$ with $\|T\| = \sup \|\tau(q)\|$. This proves the first part of the representation theorem:

10.2. Theorem. *If T is a bounded linear operator from a Banach space E to $BC(Q)$, then there exists a unique bounded continuous map τ of Q into E^* (with the E-topology) such that*

$$(1) \qquad [T(x)](q) = [\tau(q)](x), \qquad x \in E,\ q \in Q;$$

$$(2) \qquad \|T\| = \sup \{\|\tau(q)\|;\ q \in Q\}.$$

Conversely, given any such τ and defining $T: E \to BC(Q)$ by (1), *one obtains a* *linear operator with norm given by* (2). *The operator T is weakly compact if and only if τ is continuous with the E^{**}-topology in E^*. The operator T is compact if and only if τ is continuous with the norm topology in E^*.*

Proof. Only the last two statements require proof. If T is weakly compact, then by Theorem 9.4, T^* is continuous with the $BC(Q)$-topology in $BC^*(Q)$ and the E^{**}-topology in E^*, and hence τ is continuous with this topology in E^*. Conversely, if τ is continuous with the E^{**}-topology in E^* and if (q_α) is a U-net in Q which therefore converges (to some point in βQ), then $(\tau(q_\alpha))$ converges in the E^{**}-topology of E^*. Theorem 9.3 implies that $(\tau(q_\alpha))$ converges quasi-uniformly on the unit sphere S of E. Equation (1) implies that $T(S)$ is quasi-equicontinuous on Q. From Theorem 7.5 we infer that $T(S)$ is a conditionally weakly compact set in $BC(Q)$; hence T is weakly compact. The proof of the compact case follows by an exactly parallel argument and is omitted.

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